Patterns with Primes

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This first appeared as an article in *Mathematics in Schools* in November 1973.
Then it was entitled *Mining for Numbers*.

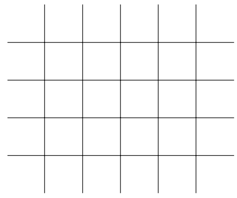
It was written after reading Chapter 9 Of Martin Gardner's
Sixth Book of Mathematical Games,
W H Freeman, 1972

The original article has been modified slightly. Also, some of the diagrams have been included on separate pages to make them suitable for the production of ohp transparencies if a classroom presentation/introduction is to be done.

A list of prime numbers is required. This can most easily be obtained from the *trol* page under Fact Sheets. The first page of that document will be more than adequate. There is also a list of primes in the *Mathematics Formulary*.

Warning!
If used as a classroom activity it can consume large quantities of squared-paper.

Once upon a time a mathematician sat listening to a boring lecture, and commenced doodling. First he sketched out a lattice like that shown on the right,



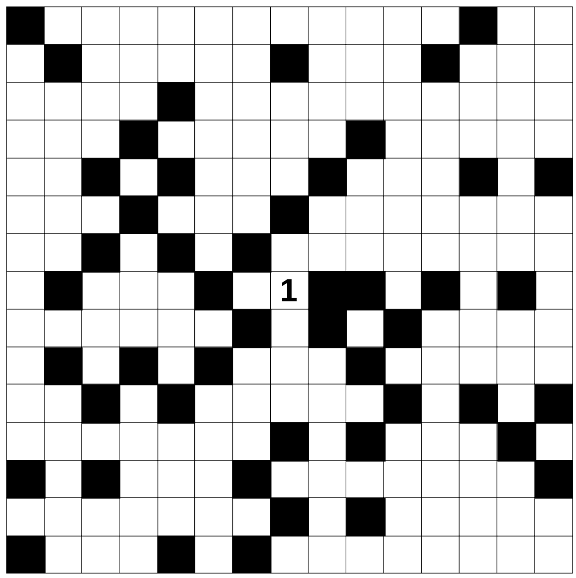
then he started to write numbers in a spiral fashion, commencing with **1**, like this

21	→	22	→	23	→	24	→	25	→	26
↑										↓
20		7	→	8	→	9	→	10		27
↑		↑						↓		↓
19		6		**1**	→	2		11		
↑		↑				↓		↓		
18		5	←	4	←	3		12		
↑								↓		
17	←	16	←	15	←	14	←	13		

“What should he do now?” he wondered. “What happened if he crossed out every prime number?” So he did that and thought that it looked like a pattern of sorts.

The next stage of course was to try it on a big scale, which he did when he got back to his office. There, he got his computer to put the first 65,000 integers in a spiral form with every prime number identified. The result of this is shown in the Gardner book mentioned.

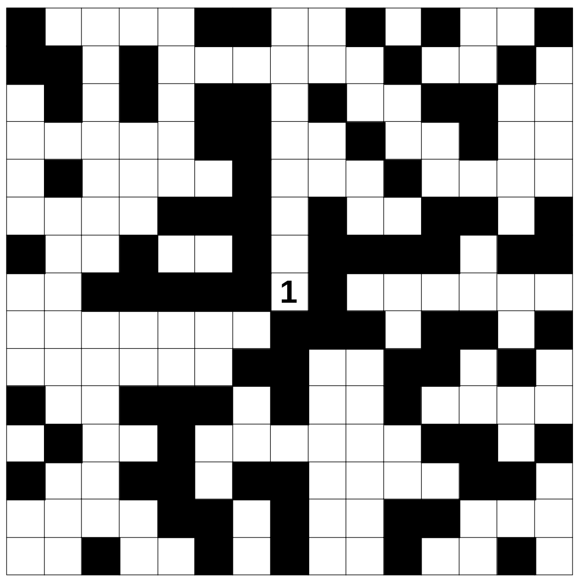
Some idea of what is happening can be gained from this diagram where the spiral goes up to 225, and every prime has been blacked out. Even in this small display, some ‘rich’ lines can be seen, that is diagonal lines which seem to contain a high proportion of primes.



Now, since we don’t expect to find any even primes (after 2) it seemed a good idea to write only the odd numbers in the lattice so it starts off

		13	→	15	→	17	→	19
		↑						↓
		11		**1**	→	3		21
↑		↑				↓		↓
35		9	←	7	←	5		23
↑								↓
33	←	31	←	29	←	27	←	25

The result of this is shown on the right, which goes up to 449. It is not surprising to find a lot more primes, though it is interesting to notice how the diagram divides into four distinct quarters.



From the foregoing, it is a short step to wondering if other ways of writing out the numbers might not provide something of interest. And that led to devising the following form of challenge.

The numbers may be written out in any order subject, of course, to the proviso that it must be a definable pattern and not based on the positions of the primes themselves! Within this array we mark off all the primes. Our “ground” is now prepared and we are ready to start “mining”.

We can drill, in a straight line only, in any direction through this ground, and all numbers falling on the line are considered to have been extracted. The task is to mine out the primes in as efficient a manner as possible.

Just how much can be done will be very dependent on the abilities of the workers. Some squared-paper and a list of primes up to maybe 200 might be enough for some. Others might be more ambitious. A definition of “efficiency” will be needed if comparisons are to be made or ratings given. Something starting like, “the line must be at least 100 (or whatever) numbers long ... “ is suggested.

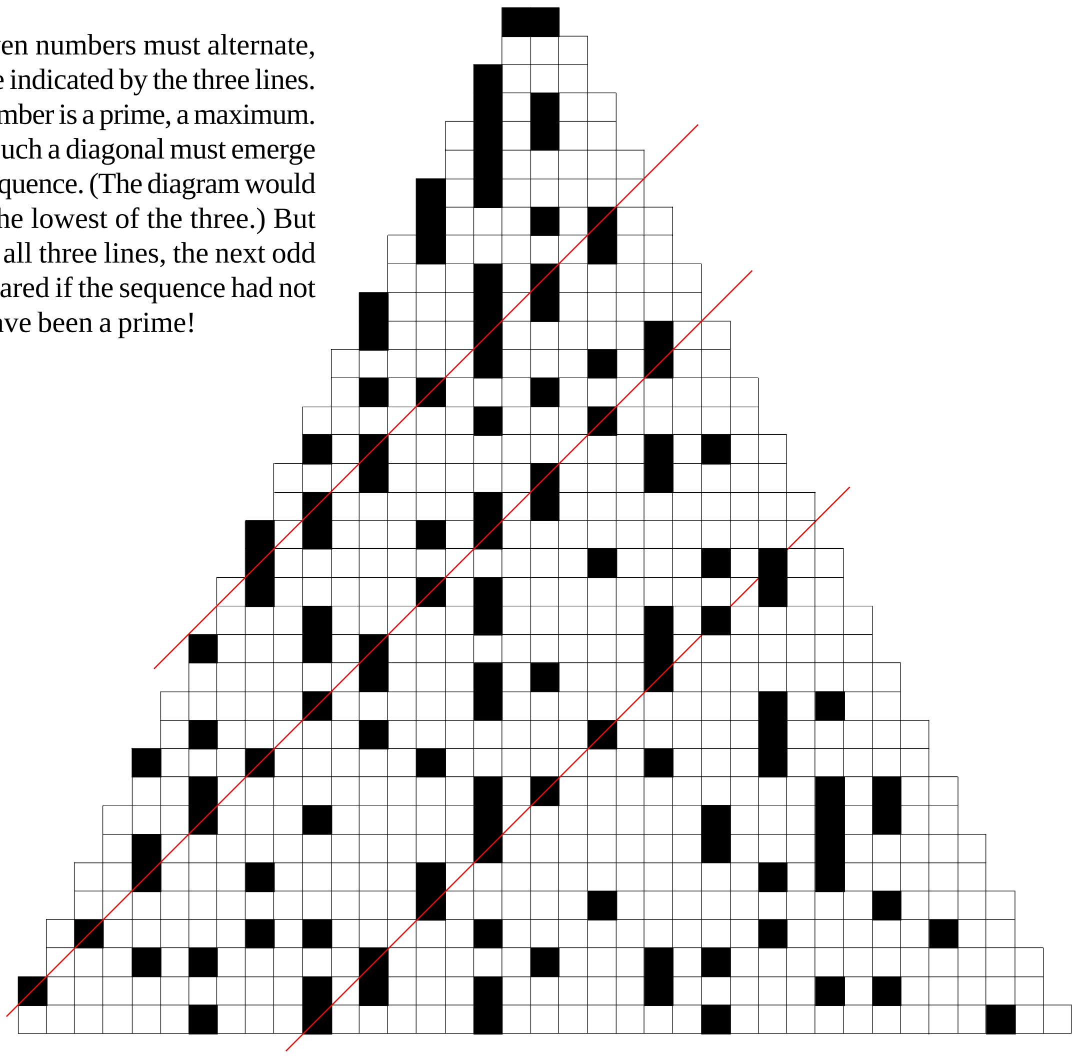
For the seriously capable the start might be the same, but the need to extend the work beyond merely drawing it all the time will soon emerge. This will require the ability to develop a formula that generates all the numbers along a given line, the drawing only being a starter to indicate which lines look to be more profitable.

The numbers could be set out like this

			1				
			2	3			
			4	5	6		
		7	8	9	10		
		11	12	13	14	15	
	16	17	18	19	20	21	
	22	23	24	25	26	27	28
29	30	31	32	etc			

where the length of each row is increased by one number each time, at alternate ends. Blacking-in the primes produces a diagram like that on the right.

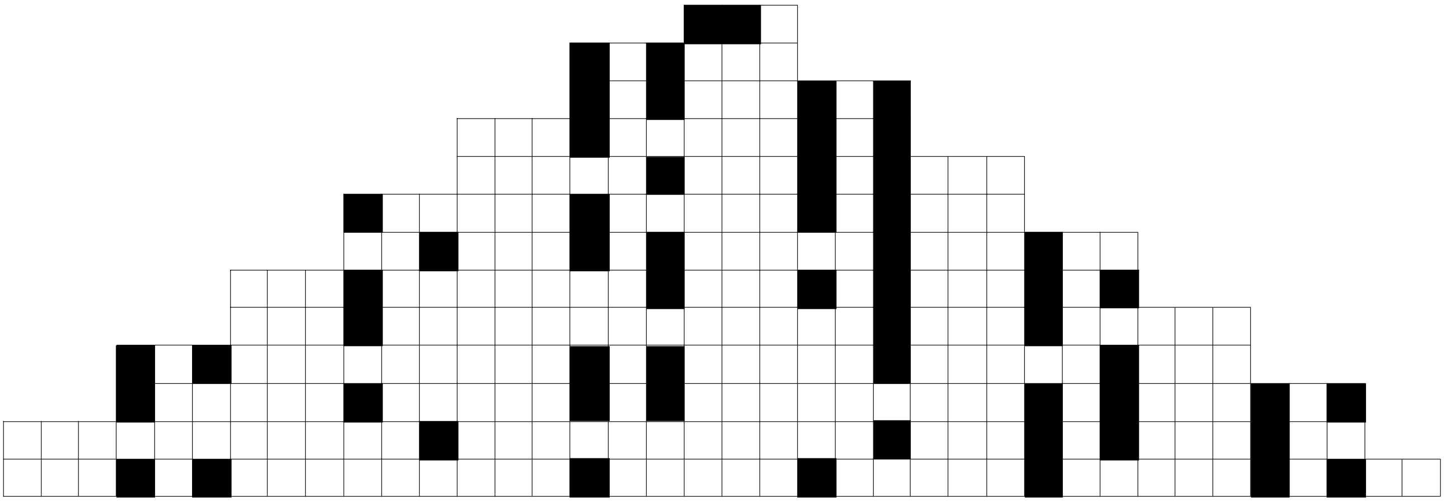
On any diagonal odd and even numbers must alternate, and the interesting ones are those indicated by the three lines. Along those lines, every other number is a prime, a maximum. Due to the shape of this ground such a diagonal must emerge on the other side and finish the sequence. (The diagram would need extending to see this for the lowest of the three.) But the fascinating thing is that, for all three lines, the next odd number which would have appeared if the sequence had not ended when it did, would not have been a prime!



The numbers could be set out like this

						2	3	4			
			5	6	7	8	9	10			
			11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29	30	31
32	33	34	35	36	etc						

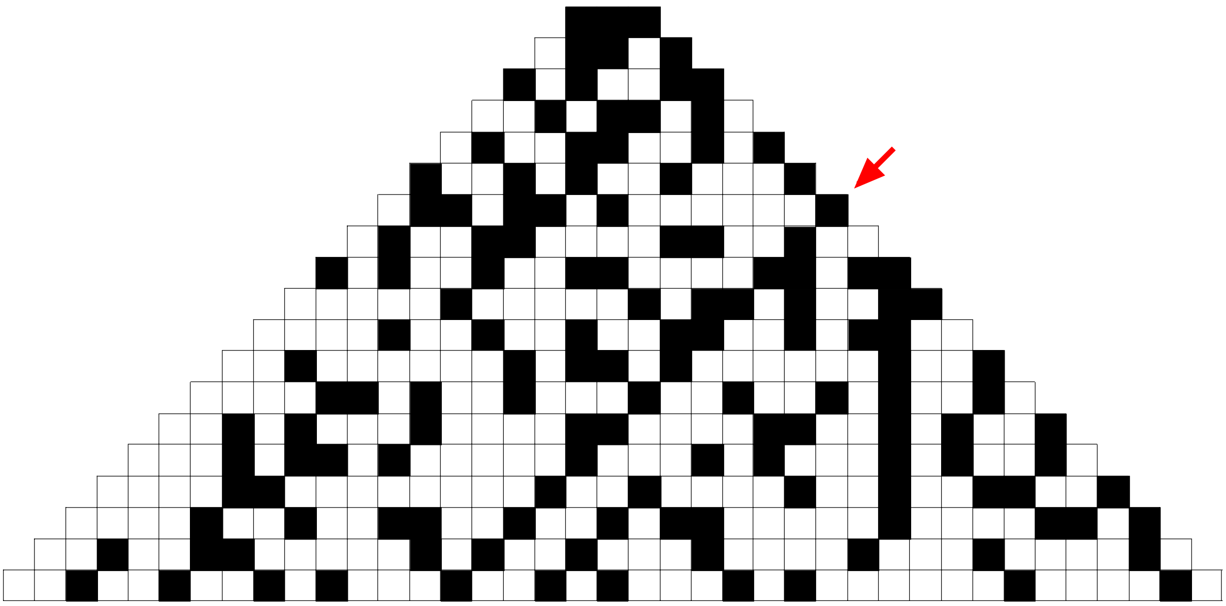
where the length of each row is increased by three numbers each time, at **alternate** ends. This produces the diagram below which, apart from its possibilities for some vertical drilling, shows very clearly how scattered the 'twin-primes' are.



Confining ourselves to the odd numbers only and setting them out like this

				3	5	7			
			9	11	13	15	17		
		19	21	23	25	27	29	31	
	33	35	37	39	41	43	45	47	49
51	53	55	57	59	61	etc			

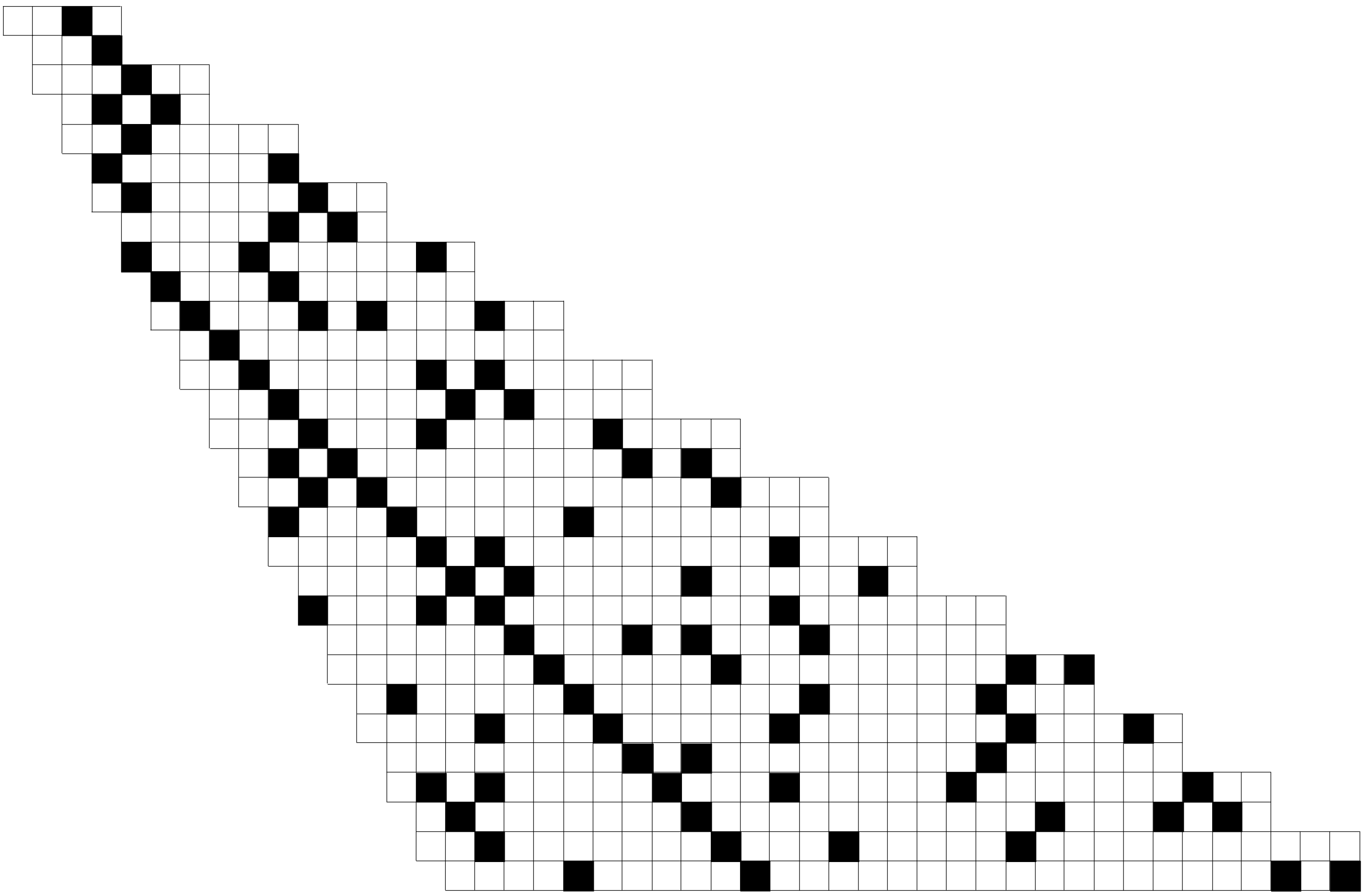
where the length of each row is increased by one number each time, at **both** ends, produces the diagram below. The diagonal line indicated by the arrow turns out to be very good. Picking out the actual numbers and extending it (algebraically) to 100 numbers reveals 74 of them to be primes.



Finally, to show how other starts and layouts may be used look at this one

27	28	29	30									
	33	32	31									
	34	35	36	37	38	39						
		44	43	42	41	40						
		45	46	47	48	49	50	51	52			
			59	58	57	56	55	54	53			
			60	61	62	63	64	65	66	67	68	69
				78	77	76	75	74	73	72	71	70
				79	80	81	82	etc				

a pattern which is much easier to write out than to explain! But moving a finger over the numbers as they are said, in order, and moving from left to right to left, soon makes it clear what is happening*. The resulting pattern of primes is shown below. The strongly marked diagonal line is indeed a rich one, yielding 81 primes when the line is extended to 100 numbers.



* Writing from left to right, and right to left, on alternate lines is described as *boustrophedron* writing. This word is derived from the Greek language and means literally “as the plough moves”.

Even so simple a layout as that shown below is capable of making clear something about the structure of the number system, and why the primes are so ‘awkwardly’ placed.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96

Marking-off (or blacking-out) all the primes produces the diagram on the right.

The second, third, fourth, and sixth columns are multiples of 2 or 3, (or both) and so those columns cannot contain a prime. Apart from the first row of course.

So primes can only be found in the first and fifth columns. Or, put another way, any prime (greater than 3) must be either 1 more or 1 less than a multiple of 6. The converse, that any multiple of 6 must have a prime either 1 before or 1 after it, is not true (see 120).

It is also of interest to note (perhaps by drawing lines through them) the positions of the multiples of 5 and 7.

1			4		6
	8	9	10		12
	14	15	16		18
	20	21	22		24
25	26	27	28		30
	32	33	34	35	36
	38	39	40		42
	44	45	46		48
49	50	51	52		54
55	56	57	58		60
	62	63	64	65	66
	68	69	70		72
	74	75	76	77	78
	80	81	82		84
85	86	87	88		90
91	92	93	94	95	96
	98	99	100		102
	104	105	106		108
	110	111	112		114
115	116	117	118	119	120
121	122	123	124	125	126
	128	129	130		132
133	134	135	136		138
139	140	141	142	143	144
145	146	147	148		150
	152	153	154	155	156
	158	159	160	161	162

Sequences and Differences

A Guide

Consider the sequence observed on page 5, which starts with 27 at the top. The interesting diagonal line, which was remarked upon, turns out to be

79 101 103 127 131 157 163 191 199 229 239 271 283 317 . . .

and we will suppose we wish to extend it.

The first thing to try is evaluating the differences between successive numbers to see if there is any pattern.
That produces this table

	79		101		103		127		131		157		163		191		199		229		239		271		283		317	. . .
1st differences		22		2		24		4		26		6		28		8		30		10		32		12		34		. . .

The differences show a mix of two sequences: 22, 24, 26, 28, 30, ... and 2, 4, 6, 8, 10, ...
Extending the table would not be difficult but, if we wish to produce a formula to generate the numbers, then it would be better to separate the two sequences by taking the alternate terms. Giving

79 103 131 163 199 239 283 . . .

and

101 127 157 191 229 271 317 . . .

We will look at only one of these for the detailed working, the other follows the same pattern.
The lower sequence and its first and second lines of differences is

	101		127		157		191		229		271		317 . . .
1st differences		26		30		34		38		42		46 . . .	
2nd differences			4		4		4		4		4 . . .		

The fact that the 2nd differences are constant, signals that the sequence is generated by a quadratic expression.
The general solution from a table having this pattern

	a_0		a_1		a_2		a_3		a_4		. . .		a_n
1st differences		b_0		b_1		b_2		b_3		. . .		b_{n-1}	
2nd differences			c		c		c		. . .		c		

is $a_n = a_0 + b_0 n + cn(n-1) \div 2$

(Notice the first term is numbered 0 and not 1)

In this particular case, $a_0 = 101$, $b_0 = 26$, $c = 4$ (and putting $c \div 2 = 2$ removes the $\div 2$ at the end)

So the required expression is $a_n = 101 + 26n + 2n(n-1)$ or $a_n = 2n^2 + 24n + 101$

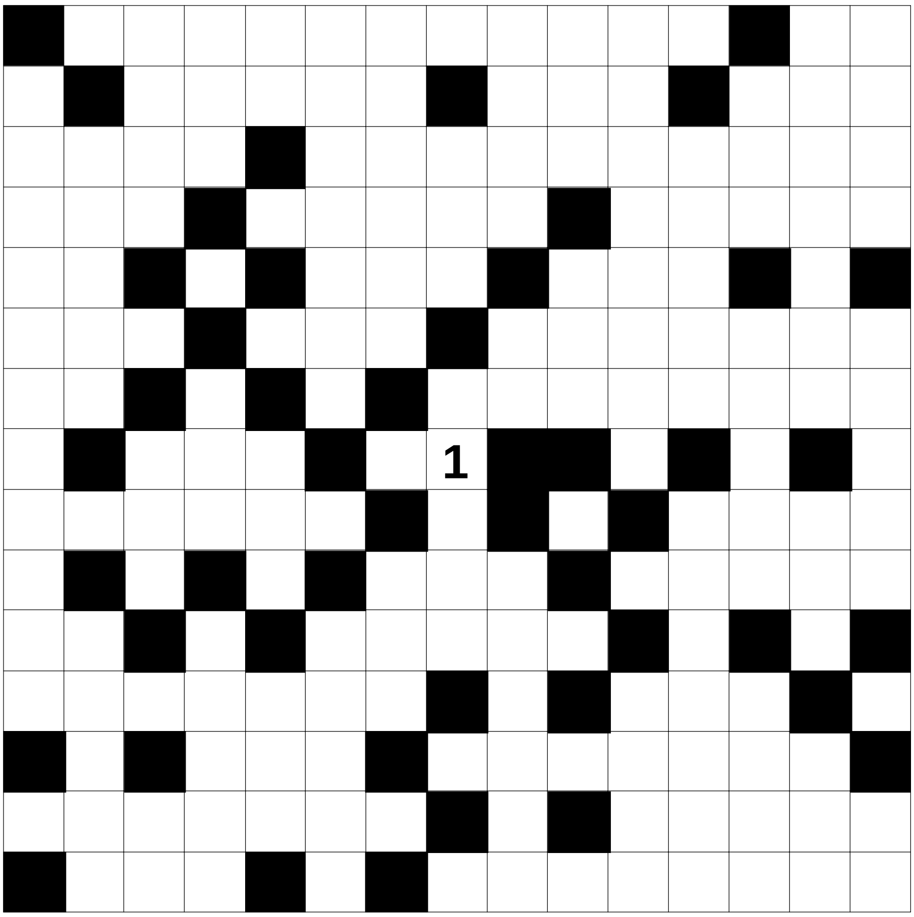
In the other case, $a_0 = 79$, $b_0 = 24$, $c = 4$ ($c \div 2 = 2$)

So the required expression is $a_n = 79 + 24n + 2n(n-1)$ or $a_n = 2n^2 + 22n + 79$

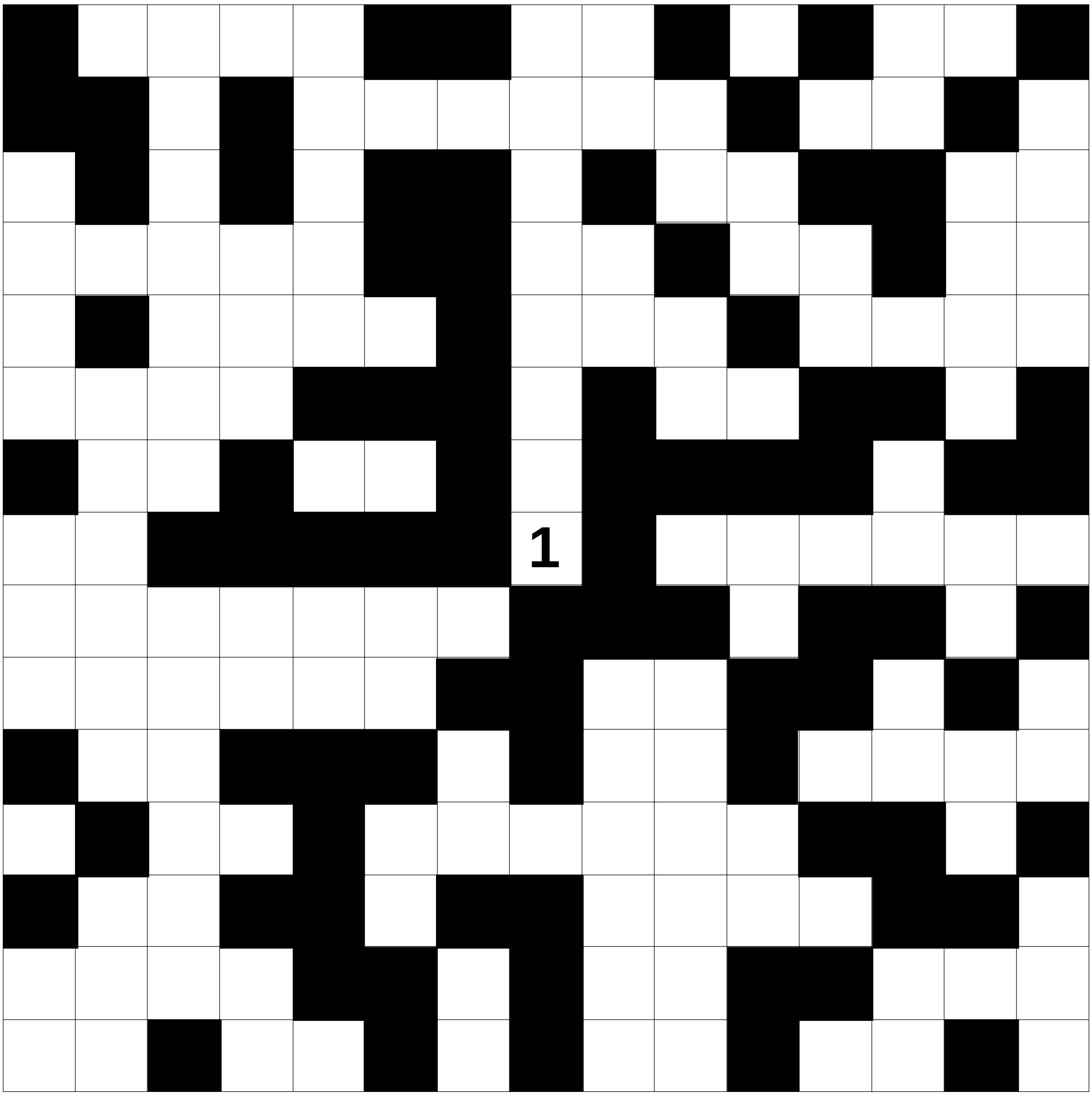
Primes are often classified by their algebraic form. The primes found in the course of this investigation are of two different forms.
One set has the form $2n^2 + 29$
which produces 29, 31, 37, 47, 61, **79**, **101**, **127**, **157**, **191**, **229**, **271**, **317** ...
The other set has the form $2n^2 - 2n + 19$
which produces 19, 23, 31, 43, 59, **79**, **103**, **131**, **163**, **199**, **239**, **283**, **331** ...
In both sets, the first five were not found in this investigation, but the use of negative values of n in the respective expressions above would reveal them.

Note. Regardless of anything that has been done here, there is NO (known) polynomial which will generate ONLY prime numbers.

43	44	45	46	47	48	49	50
42	21	22	23	24	25	26	51
41	20	7	8	9	10	27	52
40	19	6	**1**	2	11	28	53
39	18	5	4	3	12	29	
38	17	16	15	14	13	30	
37	36	35	34	33	32	31	



41	43	45	47	49	51
39	13	15	17	19	53
37	11	**1**	3	21	55
35	9	7	5	23	
33	31	29	27	25	



2 3

4 5 6

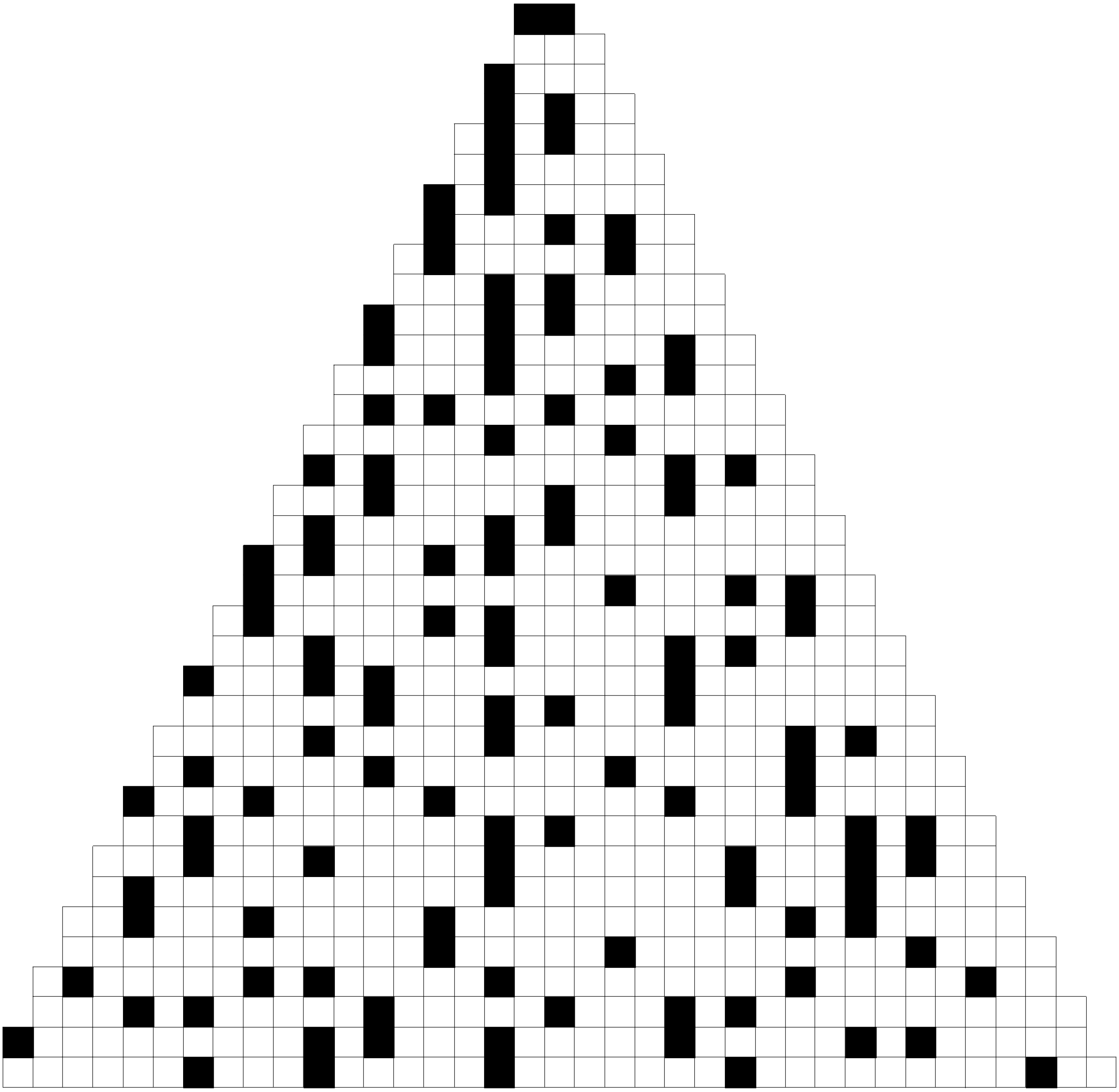
7 8 9 10

11 12 13 14 15

16 17 18 19 20 21

22 23 24 25 26 27 28

29 30 31 32 33 34 etc



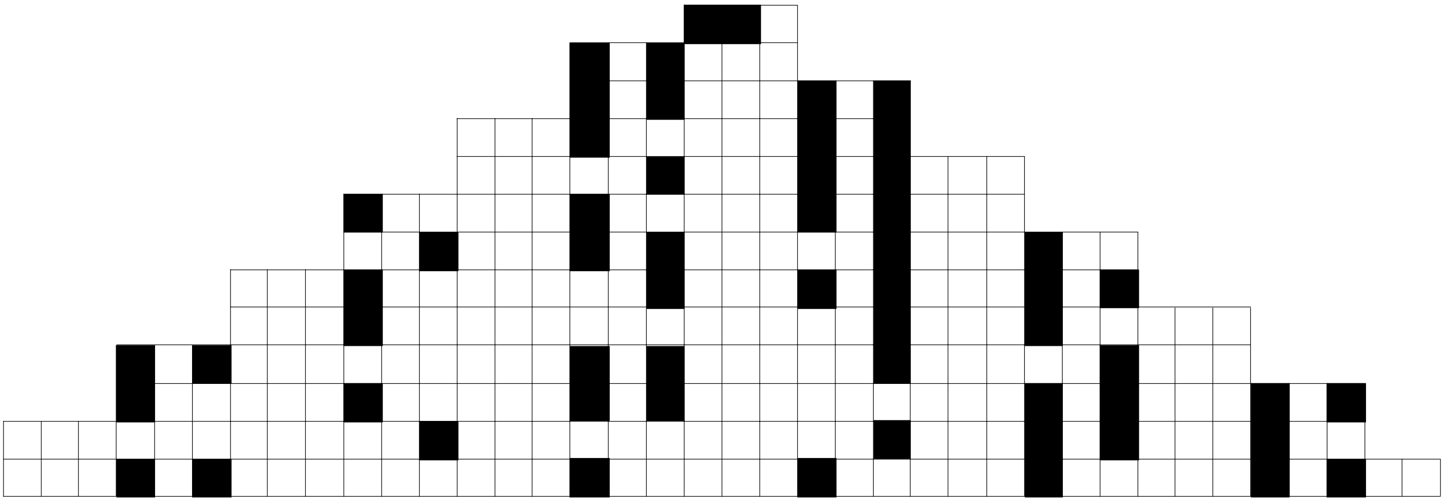
2 3 4

5 6 7 8 9 10

11 12 13 14 15 16 17 18 19

20 21 22 23 24 25 26 27 28 29 30 31

32 33 34 35 36 37 38 39 40 41 42 43 44 etc



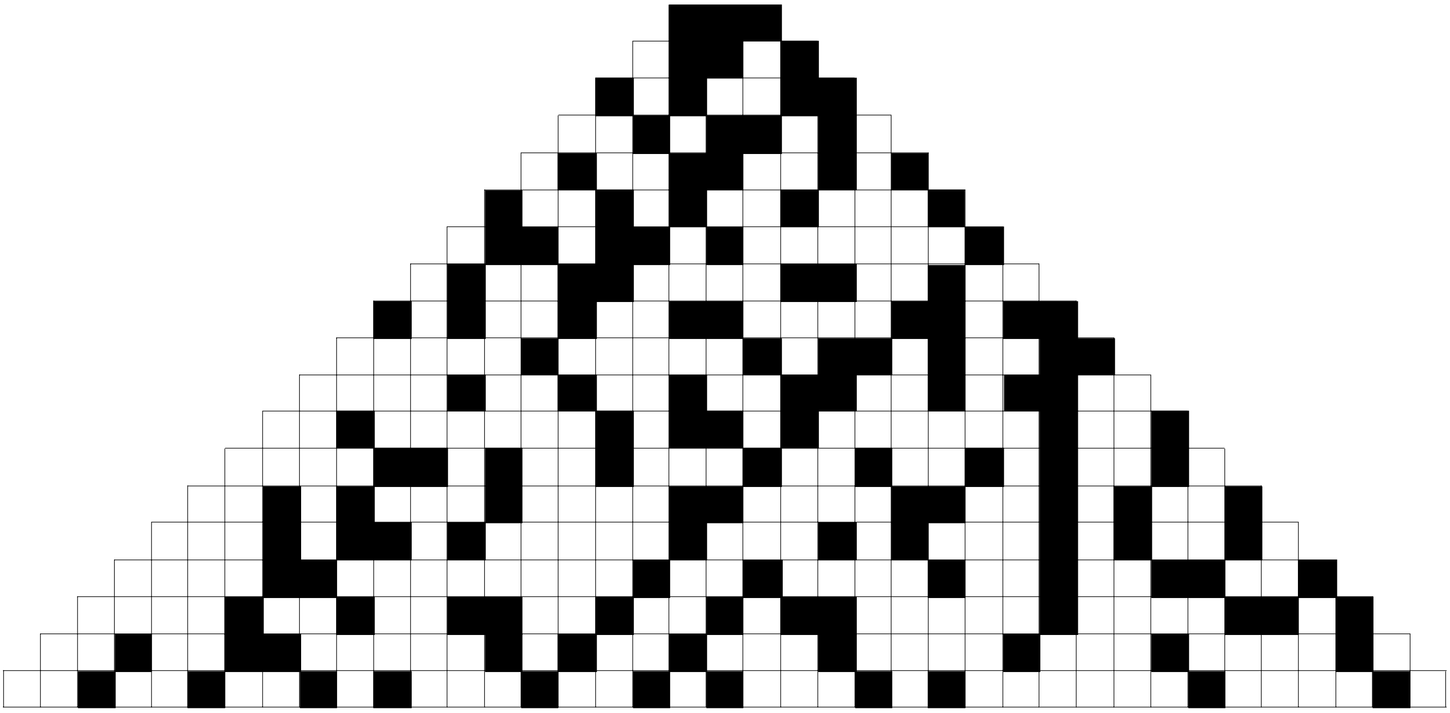
3 5 7

9 11 13 15 17

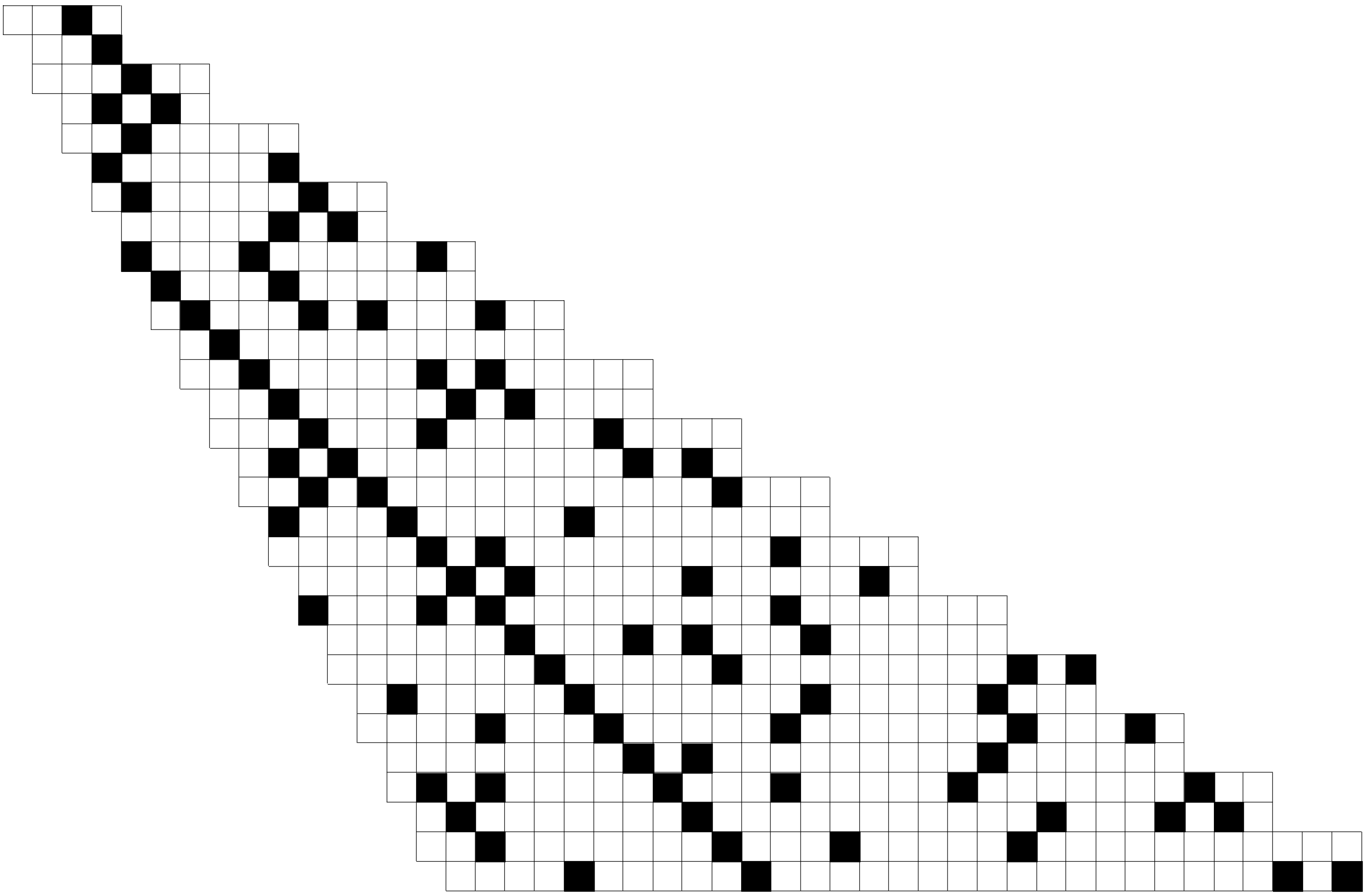
19 21 23 25 27 29 31

33 35 37 39 41 43 45 47 49

51 53 55 57 59 61 63 etc



27	28	29	30									
	32	33	31									
	34	35	36	37	38	39						
		44	43	42	41	40						
		45	46	47	48	49	50	51	52			
			59	58	57	56	55	54	53			
			60	61	62	63	64	65	66	67	68	69
				78	77	76	75	74	73	72	71	70
					79	80	81	etc				



1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102

1			4		6
	8	9	10		12
	14	15	16		18
	20	21	22		24
25	26	27	28		30
	32	33	34	35	36
	38	39	40		42
	44	45	46		48
49	50	51	52		54
55	56	57	58		60
	62	63	64	65	66
	68	69	70		72
	74	75	76	77	78
	80	81	82		84
85	86	87	88		90
91	92	93	94	95	96
	98	99	100		102